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A Positive Sum from Summability Theory

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New proofs are given for an inequality of Lorentz and Zeller which is shown to imply other inequalities which may be useful in summability theory.

1. INTRODUCTION

Lorentz and Zeller [8] proved the inequality

$$\sum_{k=0}^{\min(m,n)} \binom{m-k+\alpha}{m-k}\binom{n-k+\alpha}{n-k}\binom{k-\alpha-2}{k} \geqslant 0, \qquad \alpha \geqslant 0 \quad (1.1)$$

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and used it to obtain a new proof of a theorem of Hardy and H. Bohr. Their proof starts with a transformation of (1.1) to give

$$\sum_{k=0}^{m} \binom{m-k+\alpha}{m-k}\binom{n-k+\alpha}{n-k}\binom{k-\alpha-2}{k}$$
$$= \sum_{k=0}^{m} \binom{n+\alpha-k}{n}\binom{m-1}{m-k} \frac{(\alpha+1)\alpha\cdots(\alpha-k+2)}{k!}, \qquad (1.2)$$

$m \leqslant n$, and then they estimate the right hand side of (1.2).

Whenever a problem dealing with sums of products of binomial coefficients arises, the sum should be translated into a hypergeometric function to see what the problem really is. For hypergeometric functions are just a canonical way of writing such sums and so two seemingly different sums can be identified if they contain the same factorials. Once this transformation has been done, there is a body of knowledge which has been developed in the past two hundred years which can be used to try to solve the problem.

The sum (1.1) can be written as

$$\sum_{k=0}^{\min(m,n)} \binom{m-k+\alpha}{m-k}\binom{n-k+\alpha}{n-k}\binom{k-\alpha-2}{k}$$
$$= \frac{(\alpha+1)_m}{m!}\frac{(\alpha+1)_n}{n!}\, {}_3F_2\left(\begin{matrix}-m, -n, -\alpha-1\\ -m-\alpha, -n-\alpha\end{matrix}; 1\right),$$

so (1.1) is equivalent to

$${}_3F_2\left(\begin{matrix}-m, -n, -\alpha-1\\ -m-\alpha, -n-\alpha\end{matrix}; 1\right) \geqslant 0, \qquad \alpha \geqslant 0, \qquad m, n = 0, 1, \ldots, \qquad (1.3)$$

where

$${}_3F_2\left(\begin{matrix}a, b, c\\ d, e\end{matrix}; 1\right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n\, n!} \qquad (1.4)$$

and $(a)_n$ is the shifted factorial which can be defined by

$$(a)_n = \begin{cases} a(a+1)\cdots(a+n-1) = \Gamma(n+a)/\Gamma(a), & n = 1, 2, \ldots, \\ 1, & n = 0. \end{cases} \qquad (1.5)$$

We will consider the more general inequality

$${}_3F_2\left(\begin{matrix}-m, -n, -\alpha-1\\ -m-\beta, -n-\gamma\end{matrix}; 1\right) \geqslant 0, \qquad m, n = 0, 1, \ldots, \qquad (1.6)$$

which is equivalent to

$$\sum_{k=0}^{\min(m,n)} \binom{m-k+\beta}{m-k}\binom{n-k+\gamma}{n-k}\binom{k-\alpha-2}{k} \geqslant 0, \quad m, n = 0, 1,\ldots \tag{1.7}$$

and prove that these inequalities hold for $0 \leqslant \alpha \leqslant \beta$, $0 \leqslant \alpha \leqslant \gamma$.

A somewhat related, but easier, question was treated by Askey [1] and Bustoz [4]. Askey proved that

$$\begin{aligned} {}_2F_1(-n, -\gamma-1; -n-\alpha; r) \geqslant 0, \qquad 0 \leqslant r \leqslant (\alpha+1)/(\gamma+1), \\ -1 < \alpha < \gamma, \end{aligned} \tag{1.8}$$

and Bustoz gave a direct proof that

$$\frac{(1-rt)^{\gamma+1}}{(1-t)^{\alpha+1}} = \sum_{n=0}^{\infty} g_n(r; \alpha, \gamma)\, t^n,$$

with $g_n(r; \alpha, \gamma) \geqslant 0$, $0 \leqslant r \leqslant (\alpha+1)/(\gamma+1)$, $-1 < \alpha < \gamma$. The idea of using generating functions to change (1.6) to an equivalent problem is the crux of our proof and one of the proofs uses recurrence relations as in Askey's proof of (1.8).

2. Main Theorem

THEOREM 1. *If* $0 \leqslant \alpha \leqslant \min(\beta, \gamma)$ *then the following results hold.*

$$\sum_{k=0}^{\min(m,n)} \binom{m-k+\beta}{m-k}\binom{n-k+\gamma}{n-k}\binom{k-\alpha-2}{k} \geqslant 0, \quad m, n = 0, 1,\ldots, \tag{2.1}$$

$${}_3F_2\left(\begin{matrix}-m, -n, -\alpha-1 \\ -m-\beta, -n-\gamma\end{matrix}; 1\right) \geqslant 0, \qquad m, n = 0, 1,\ldots, \tag{2.2}$$

$$\frac{(1-rs)^{\alpha+1}}{(1-r)^{\beta+1}(1-s)^{\gamma+1}} = \sum_{m,n=0}^{\infty} h(m, n; \alpha, \beta, \gamma)\, r^m s^n, \tag{2.3}$$

with $h(m, n; \alpha, \beta, \gamma) \geqslant 0$, $m, n = 0, 1,\ldots.$

The condition $\alpha \leqslant \min(\beta, \gamma)$ is a necessary condition for any of these results to hold.

Remark. The ${}_3F_2$ in (2.2) is assumed to terminate with $\min(m+1, n+1)$ terms even if β or γ is an integer.

Proof. The equivalence of (2.1) and (2.2) is true when $\beta > -1, \gamma > -1$, for

$$\sum_{k=0}^{\min(m,n)} \binom{m-k+\beta}{m-k}\binom{n-k+\gamma}{n-k}\binom{k-\alpha-2}{k}$$
$$= \frac{(\beta+1)_m}{m!}\frac{(\gamma+1)_n}{n!}\,{}_3F_2\left(\begin{matrix}-m, -n, -\alpha-1\\ -m-\beta, -n-\gamma\end{matrix}; 1\right). \tag{2.4}$$

Multiply the right hand side of (2.4) by $r^m s^n$, sum on m and n, and reverse the order of summation (i.e., sum last on k, the summation index of the ${}_3F_2$) to obtain

$$\sum_{m,n=0}^{\infty} \frac{(\beta+1)_m}{m!}\frac{(\gamma+1)_n}{n!}\,{}_3F_2\left(\begin{matrix}-m, -n, -\alpha-1\\ -m-\beta, -n-\gamma\end{matrix}; 1\right) r^m s^n$$
$$= \frac{(1-rs)^{\alpha+1}}{(1-r)^{\beta+1}(1-s)^{\gamma+1}} = (1-r)^{\alpha-\beta}(1-s)^{\alpha-\gamma}\frac{(1-rs)^{\alpha+1}}{(1-r)^{\alpha+1}(1-s)^{\alpha+1}}. \tag{2.5}$$

The two factors $(1-r)^{\alpha-\beta}$ and $(1-s)^{\alpha-\gamma}$ have nonnegative power series coefficients when $\beta \geqslant \alpha$ and $\gamma \geqslant \alpha$. So to prove Theorem 1 it is sufficient to prove it when $\alpha = \beta = \gamma$. It is also sufficient to prove it when $0 < \alpha < 1$, for the case $\alpha = 0$ is

$$\frac{(1-rs)}{(1-r)(1-s)} = \frac{2-r-s}{(1-r)(1-s)} - 1 = 1 + \sum_{n=1}^{\infty} r^n + \sum_{n=1}^{\infty} s^n$$

and the case $\alpha \geqslant 1$ can be reduced to the case $0 \leqslant \alpha < 1$ by

$$\frac{(1-rs)^{\alpha+1}}{(1-r)^{\alpha+1}(1-s)^{\alpha+1}} = \frac{(1-rs)^{\alpha-\lfloor\alpha\rfloor+1}}{(1-r)^{\alpha-\lfloor\alpha\rfloor+1}(1-s)^{\alpha-\lfloor\alpha\rfloor+1}}\cdot\frac{(1-rs)^{\lfloor\alpha\rfloor}}{(1-r)^{\lfloor\alpha\rfloor}(1-s)^{\lfloor\alpha\rfloor}},$$

where $\lfloor\alpha\rfloor$ is the greatest integer less than or equal to α.

The most important step in the Lorentz–Zeller proof of (1.1) was the transformation (1.2). There are many transformations of hypergeometric functions dating back to Euler for the ordinary ${}_2F_1$ and to Kummer for the ${}_3F_2$. Kummer [7, p. 172] stated the transformation

$${}_3F_2\left(\begin{matrix}a, b, c\\ d, e\end{matrix}; 1\right)$$
$$= \frac{\Gamma(e)\,\Gamma(d+e-a-b-c)}{\Gamma(e-c)\,\Gamma(d+e-a-b)}\,{}_3F_2\left(\begin{matrix}d-a, d-b, c\\ d, d+e-a-b\end{matrix}; 1\right). \tag{2.6}$$

See Hardy [6] for a simple proof. Thomae [9] obtained many other transformations, one of them being

$$ {}_3F_2\left({a, b, c \atop d, e}; 1\right) = \frac{\Gamma(d)\,\Gamma(e)\,\Gamma(s)}{\Gamma(a)\,\Gamma(s+b)\,\Gamma(s+c)}\, {}_3F_2\left({d-a, e-a, s \atop s+b, s+c}; 1\right) \tag{2.7} $$

where $s = d + e - a - b - c$. This can be obtained by iterating (2.6). Whipple [10] gave a systematic treatment of this type of transformation and a summary of Whipple's classification is given in Bailey [3]. Apply (2.7) to (2.2) when $\alpha = \beta = \gamma$ with $a = -\alpha - 1$ to obtain

$$ \begin{aligned} &{}_3F_2\left({-m, -n, -\alpha-1 \atop -m-\alpha, -n-\alpha}; 1\right) \\ &= \frac{\Gamma(-m-\alpha)\,\Gamma(-n-\alpha)\,\Gamma(1-\alpha)}{\Gamma(-\alpha-1)\,\Gamma(1-m-\alpha)\,\Gamma(1-n-\alpha)} \\ &\quad \cdot {}_3F_2\left({1-m, 1-n, 1-\alpha \atop 1-m-\alpha, 1-n-\alpha}; 1\right) \\ &= \frac{\alpha(\alpha+1)}{(m+\alpha)(n+\alpha)}\, {}_3F_2\left({1-m, 1-n, 1-\alpha \atop 1-m-\alpha, 1-n-\alpha}; 1\right). \end{aligned} \tag{2.8} $$

The right hand side of (2.8) is clearly positive when $0 < \alpha < 1$, for $(1-\alpha)_k > 0$ and

$$ \frac{(1-m)_k}{(1-m-\alpha)_k} > 0, k = 0, 1,..., m-1, \frac{(1-n)_k}{(1-n-\alpha)_k} > 0, k = 0, 1,..., n-1 $$

and $(1-m)_k(1-n)_k = 0$ for $k \geqslant m$ or $k \geqslant n$.

Let $m = 1$ in (2.2) to obtain

$$ {}_3F_2\left({-1, -n, -\alpha-1 \atop -1-\beta, -n-\gamma}; 1\right) = 1 - \frac{(\alpha+1)\,n}{(1+\beta)(n+\gamma)}. $$

Then let $n \to \infty$ to see that $\alpha \leqslant \beta$ is a necessary condition. By symmetry $\alpha \leqslant \gamma$ is also necessary.

This completes one proof of Theorem 1. The Lorentz–Zeller proof can be started by using the Kummer–Thomae–Whipple transformation formulas, since (1.2) is one of these formulas after a suitable limit has been taken in the hypergeometric function transformation. Details will not be given since it is clearly preferable to use the above proof which shows positivity by writing the sum as a new sum of strictly positive terms.

3. Second Proof

There is another proof which is worth giving for two reasons. First, it does not distinguish the cases $0 < \alpha < 1$ from the other cases, and second, it gives an interesting absolutely monotonic function (one whose power series coefficients are nonnegative). For simplicity we will only treat the case $\alpha = \beta = \gamma$.

Form the generating function

$$F_m(r) = \sum_{n=0}^{\infty} \frac{(\alpha+1)_n}{n!} r^n \, {}_3F_2\left(\begin{matrix} -m, -n, -\alpha-1 \\ -m-\alpha, -n-\alpha \end{matrix}; 1\right)$$

$$= (1-r)^{-\alpha-1} {}_2F_1\left(\begin{matrix} -\alpha-1, -m \\ -m-\alpha \end{matrix}; r\right)$$

$$= {}_2F_1\left(\begin{matrix} -\alpha-1, -\alpha \\ -m-\alpha \end{matrix}; \frac{-r}{1-r}\right),$$

by Pfaff's formula

$${}_2F_1(a, b; c; x) = (1-x)^{-a} {}_2F_1(a, c-b; c; -x/(1-x)).$$

See [5, 2.1.4(22)]. So the problem reduces to showing that

$${}_2F_1(-\alpha-1, -\alpha; -m-\alpha; -r/(1-r))$$

is absolutely monotonic when $\alpha > 0$.

Using the contiguous relation [5, 2.8(35)]

$$(a-c+1)\,{}_2F_1(a, b; c; x) = a\,{}_2F_1(a+1, b; c; x) - (c-1)\,{}_2F_1(a, b; c-1; x)$$

we get

$$(m+\alpha+1) F_{m+1}(r) = mF_m(r) + (\alpha+1)\,{}_2F_1(-\alpha, -\alpha; -m-\alpha; -r/(1-r)). \tag{3.1}$$

It is clear that the coefficient of r^n in ${}_2F_1(-\alpha, -\alpha; -m-\alpha; -r/(1-r))$ is nonnegative for $0 \leqslant n \leqslant m+1$. Let us denote ${}_3F_2\left(\begin{smallmatrix} -m, -n, -\alpha-1 \\ -m-\alpha, -n-\alpha \end{smallmatrix}; 1\right)$ by $f_{m,n}$ and the coefficient of r^n in the power series for ${}_2F_1(-\alpha, -\alpha; -m-\alpha; -r/(1-r))$ by $c_{m,n}$. Then (3.1) gives

$$(m+\alpha+1)(\alpha+1)_n f_{m+1,n} = m(\alpha+1)_n f_{m,n} + (\alpha+1)\, n! c_{m,n}. \tag{3.2}$$

Clearly $f_{0,n} \geqslant 0$ and $f_{n,0} \geqslant 0$ for $n = 0, 1, \ldots$. Assume that $f_{j,k} \geqslant 0$ for $\min(j, k) \leqslant n$. If we can show that this implies $f_{n+1,n+1} \geqslant 0$ then (3.2) implies $f_{m,n+1} \geqslant 0$, $m = n + 2, n + 3, \ldots$, and by symmetry, $f_{n+1,m} \geqslant 0$, $m = n + 2, n + 3, \ldots$. Then the proof continues by induction. But

$$(n + \alpha + 1)(\alpha + 1)_{n+1} f_{n+1,n+1} = n(\alpha + 1)_{n+1} f_{n,n+1} + (\alpha + 1)(n + 1)!\, c_{n,n+1}$$

and both $f_{n,n+1} \geqslant 0$ and $c_{n,n+1} \geqslant 0$, so $f_{n+1,n+1} \geqslant 0$.

4. Open Problems and Related Results

When $\alpha = \beta > -1$ the condition $\gamma \geqslant 0$ is necessary. This follows from the special case $m = n = 1$. This case gives the condition

$$1 - \frac{\alpha + 1}{(\beta + 1)(\gamma + 1)} \geqslant 0 \tag{4.1}$$

which, when $\beta > -1$ and $\gamma > -1$, is equivalent to

$$\beta + \gamma + \beta\gamma \geqslant \alpha.$$

It is reasonable to assume $\alpha, \beta, \gamma > -1$ but so far the cases when one or more of these numbers is negative have not been done.

There have been a few results similar to Theorem 1 in the recent literature. Askey and Gasper [2] proved that

$$\begin{aligned} &{}_3F_2\left(\begin{matrix} -k, -m, -n \\ (-k - m - n - \alpha)/2, (-k - m - n - \alpha + 1)/2 \end{matrix}; 1\right) \geqslant 0, \\ &\qquad \alpha \geqslant (-5 + \sqrt{17})/2 \end{aligned} \tag{4.2}$$

and that this inequality fails when $k = m = n = 1$, $\alpha < (-5 + \sqrt{17})/2$. The sum of the denominator parameters is of the of the same size as the sum of the numerator parameters and this is also the case in (2.2). This must have some significance, but exactly what is not clear. It is unlikely that a complete answer will ever be given to the question of the positivity of general ${}_3F_2$'s, but these results, and the further results in Askey–Gasper [2] for ${}_3F_2$'s of a different nature, give some indication of the type of results that can be expected. It is likely that further problems of this type will arise, and a few methods now exist for treating them.

Note added in proof. The last reference in the body of this paper is incorrect. It should refer to: R. Askey and G. Gasper, Jacobi polynomial expansions of Jacobi polynomials with nonnegative coefficients, *Proc. Cambridge Philos. Soc.* **70** (1971), 243–255.

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